Numer. Math. (2006) 104: 539–559
DOI 10.1007/s00211-006-0015-4

Numerische Mathematik

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One-sided stability and convergence of the Nessyahu–Tadmor scheme

Received: 18 September 2005 / Revised: 12 April 2006 / Published online: 13 September 2006

Abstract Non-oscillatory schemes are widely used in numerical approximations of nonlinear conservation laws. The Nessyahu–Tadmor (NT) scheme is an example of a second order scheme that is both robust and simple. In this paper, we prove a new stability property of the NT scheme based on the standard minmod reconstruction in the case of a scalar strictly convex conservation law. This property is similar to the One-sided Lipschitz condition for first order schemes. Using this new stability, we derive the convergence of the NT scheme to the exact entropy solution without imposing any nonhomogeneous limitations on the method. We also derive an error estimate for monotone initial data.

AMS Subject Classification Primary 65M15 · Secondary 65M12

1 Introduction

We are interested in the scalar hyperbolic conservation law

$$\begin{cases} u_t + f(u)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u^0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where f is a given flux function. In recent years, there has been enormous activity in the development of the mathematical theory and in the construction of numerical methods for (1). Even though the existence-uniqueness theory is complete, there

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are many numerically efficient methods for which the questions of convergence and error estimates are still open. For example, there are many second or higher order non-oscillatory schemes based on minmod limiters which are numerically robust but theoretical results about convergence or error estimates are still missing [6–9,16,18,21]. Usually second order schemes are constructed to be total variation diminishing (TVD) but that property only guarantees the convergence of such schemes to a weak solution, see [10]. No property was known that implies convergence of such schemes to the entropy solution even in the case of a genuinely nonlinear scalar conservation law. The usual approach is to try to prove a single cell entropy inequality which usually leads to additional nonhomogeneous limitations on a second order scheme in order to fit it into the existing convergence theory. There are few results on convergence of non-oscillatory second order schemes which do not require nonhomogeneous limitations and we are going to mention them here. LeFloch and Liu in [11] consider piecewise smooth data and prove a different entropy inequality in different monotonicity regions of the numerical solution. Their result is valid for a specific second order upwind scheme and it may work for other schemes but the conditions are hard to check, the NT schemes does not fit into their framework, and there is little hope to prove any error estimates with that approach. In [22,23], Yang reduces the convergence of a special type second order scheme to a convergence of that scheme for a Riemann problem. Again, that type of argument has no potential for any error estimates. Finally, Lions and Souganidis [12] develop a convergence theory for second order schemes for scalar convex conservation laws and Hamilton–Jacobi equations. Unfortunately, their results for conservation laws do not hold for any of the explicit second order schemes used in practice because of the very strong restriction imposed on the CFL condition, see [12]. The main reason for such difficulties is hidden in the fact that besides a TVD property very little was known for non-oscillatory schemes because they use nonlinear limiters such as Minmod. This is in contrast to the theory for first order schemes where in the convex case there are many different approaches. For example, Tadmor's dual approach based on Lip+ stability [19] and the Kruzkov-Kuznetsov argument based on an entropy diminishing property [2,1,17]. In our previous work [13], in the case of a linear flux, we derive a new stability result for a generic second order scheme (central or upwind) based on the Minmod limiter. Here, we prove the one-sided analog of this result for the NT schemes in the case of any scalar conservation law with a strictly convex flux. This new property, to the best of our knowledge, is the first one-sided stability result for a second order scheme. We use that result to prove convergence of the NT scheme to the unique entropy solution without imposing any nonhomogeneous limitation on the method. This stability result and our results in [14] imply an error estimate in the case of monotone initial data. The question of a general error estimate framework based on the new stability will be addressed elsewhere. All results in this paper are also valid for the non-staggered version of the NT scheme based on the Minmod limiter given in [9].

The paper is organized as follows. In Sect. 2, we describe the staggered NT scheme. In Sect. 3, we present our main result: a new one-sided stability property of the NT scheme. Then, we use that property to prove the convergence of the scheme to the entropy solution and derive an error estimate for monotone initial data in Sect. 4.

2 Non-oscillatory central schemes

In this section, we are concerned with second order non-oscillatory central differencing approximations to the scalar conservation law

$$u_t + f(u)_x = 0. \tag{2}$$

The prototype of all such schemes is the staggered Nessyahu–Tadmor (NT) scheme [18]. We limit our attention to the staggered NT scheme but all results in this paper are valid for the corresponding non-staggered version in [9]. We now recall the basic step in the NT scheme [18]. Let $v(x,t)$ be an approximate solution to (2), and assume that the space mesh Δx and the time mesh Δt are uniform. Let $x_j := j\Delta x$, $j \in \mathbb{Z}$, $\lambda := \Delta t/\Delta x$ and

$$v_j(t) := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} v(x,t)\,\mathrm{d}x \tag{3}$$

be the average of v at time t over $(x_{j-1/2}, x_{j+1/2})$. Let us assume that $v(\cdot,t)$ is a piecewise linear function, and it is linear on the intervals $(x_{j-1/2}, x_{j+1/2})$, $j \in \mathbb{Z}$, of the form

$$v(x,t) = L_j(x,t) := v_j(t) + (x - x_j)\frac{1}{\Delta x}v'_j, \quad x_{j-1/2} < x < x_{j+1/2}, \tag{4}$$

where $\frac{1}{\Delta x}v'_j$ is the numerical derivative of v which is yet to be determined. Integration of (2) over the staggered space-time cell $(x_j, x_{j+1}) \times (t, t+\Delta t)$ yields

$$\begin{aligned} v_{j+1/2}(t+\Delta t) = {} & \frac{1}{\Delta x}\left(\int_{x_j}^{x_{j+1/2}} L_j(x,t)\,\mathrm{d}x + \int_{x_{j+1/2}}^{x_{j+1}} L_{j+1}(x,t)\,\mathrm{d}x\right) \\ & - \frac{1}{\Delta x}\left(\int_t^{t+\Delta t} f(v(x_{j+1},\tau))\,\mathrm{d}\tau - \int_t^{t+\Delta t} f(v(x_j,\tau))\,\mathrm{d}\tau\right). \end{aligned} \tag{5}$$

The first two integrals on the right of (5) can be evaluated exactly. Moreover, if the CFL condition

$$\lambda \max_{x_j \le x \le x_{j+1}} |f'(v(x,t))| \le \frac{1}{2}, \quad j \in \mathbb{Z}, \tag{6}$$

is met, then the last two integrants on the right of (5) are smooth functions of τ. Hence, they can be integrated approximately by the midpoint rule with third order local truncation error. Note that, in the case of zero slopes $\frac{1}{\Delta x}v'_j$ and $\frac{1}{\Delta x}v'_{j+1}$, the time integration is exact for any flux f. Thus, following [18], we arrive at

$$\begin{aligned} v_{j+1/2}(t+\Delta t) = {} & \frac{1}{2}(v_j(t) + v_{j+1}(t)) + \frac{1}{8}\left(v'_j - v'_{j+1}\right) \\ & - \lambda\left(f\left(v\left(x_{j+1}, t + \frac{\Delta t}{2}\right)\right) - f\left(v\left(x_j, t + \frac{\Delta t}{2}\right)\right)\right). \end{aligned} \tag{7}$$

By Taylor expansion and the conservation law (2), we obtain

$$v\left(x_j, t+\frac{\Delta t}{2}\right) = v_j(t) - \frac{1}{2}\lambda f_j', \tag{8}$$

where $\frac{1}{\Delta x} f_j'$ stands for an approximate numerical derivative of the flux $f(v(x = x_j, t))$. The following choices are widely used as approximations of the numerical derivatives (we drop t to simplify the notation)

$$v_j' = \mathrm{m}(v_{j+1} - v_j, v_j - v_{j-1}), \tag{9}$$

$$f_j' = \mathrm{m}(f(v_{j+1}) - f(v_j), f(v_j) - f(v_{j-1})) \quad \text{or} \quad f_j' = f'(v_j^n)\, v_j', \tag{10}$$

where $\mathrm{m}(a, b)$ stands for the standard minmod limiter

$$\mathrm{m}(a, b) \equiv \mathrm{MinMod}(a, b) := \frac{1}{2}(\mathrm{sgn}(a) + \mathrm{sgn}(b)) \cdot \min(|a|, |b|). \tag{11}$$

Using the approximate slopes (9) and flux derivatives (10), we construct a family of central schemes in the predictor–corrector form

$$\begin{aligned} v\left(x_j, t+\frac{\Delta t}{2}\right) &= v_j(t) - \frac{1}{2}\lambda f_j', \\ v_{j+1/2}(t+\Delta t) &= \frac{1}{2}\left(v_j(t) + v_{j+1}(t)\right) + \frac{1}{8}(v_j' - v_{j+1}') \\ &\quad - \lambda\left(f\left(v\left(x_{j+1}, t+\frac{\Delta t}{2}\right)\right) - f\left(v\left(x_j, t+\frac{\Delta t}{2}\right)\right)\right), \end{aligned} \tag{12}$$

where we start with $v_j(0) := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,\mathrm{d}x$. Note that, this is the description of the NT scheme when we compute the staggered averages from the averages on the regular grid. The other step, from the staggered averages to the averages on the regular grid, is completely analogous – we have to shift the index j to $j + 1/2$ everywhere. Therefore, we compute averages on two staggered uniform partitions of the real line: (i) all intervals $I_j := (x_{j-1/2}, x_{j+1/2})$, $j \in \mathbb{Z}$ for time $t = 2n\Delta t$, $n = 0, 1, \ldots$; (ii) all intervals $J_j := (x_j, x_{j+1})$, $j \in \mathbb{Z}$ for time $t = (2n+1)\Delta t$, $n = 0, 1, \ldots$

3 One-sided stability of the NT scheme

In this section we present the main result of the paper: a new stability result for the NT scheme. We will consider only the case of even n with the case of odd n being analogous. Let us denote the numerical solution of the NT scheme at time step $t_n = n\Delta t$ with v^n, $v^n := v(\cdot, t_n)$, and its cell averages with v_j^n, where the cell averages of v^0 are equal to the cell averages of the initial condition u^0: $v_j^0 := u_j^0$, $j \in \mathbb{Z}$. We define the numerical solution $v(\cdot, t_n)$ to be a linear function on each cell $I_j := (x_{j-1/2}, x_{j+1/2})$

$$v^n|_{I_j} = v_j^n + (x - x_j)\frac{1}{\Delta x}\mathrm{m}\left(v_{j+1}^n - v_j^n, v_j^n - v_{j-1}^n\right). \tag{13}$$

We are going to use the notation: v_j^n for the cell averages at time t_n, $v_{j+1/2}^{n+1}$ for the cell averages at time t_{n+1}, and $a(u) := f'(u)$ for the derivative of the flux. Then, the averages at time t_{n+1} are given by

$$v_{j+1/2}^{n+1} = \frac{1}{2}\left(v_j^n + v_{j+1}^n\right) + \frac{1}{8}\left(v_j' - v_{j+1}'\right) - \lambda\left[f\left(v_{j+1}^{n+1/2}\right) - f\left(v_j^{n+1/2}\right)\right], \tag{14}$$

where v_j' is given by (9), $v_j^{n+1/2} = v_j^n - (\lambda/2) f_j'$, and f_j' is one of the flux approximations given in (10).

Denote the *new* jumps at time t_{n+1} with $\delta_j^{n+1} := v_{j-1/2}^{n+1} - v_{j-3/2}^{n+1}$, the *old* jumps (at time t_n) with $\delta_j^n := v_j^n - v_{j-1}^n$, and let $\|a\|_\infty$ be the maximum speed of propagation

$$\|a\|_\infty := \max_{|w|\le \|u^0\|_{L^\infty}} |f'(w)|. \tag{15}$$

With this notation we have the following theorem which is our main result.

Theorem 1 *Let $u^0 \in L^\infty(\mathbb{R})$ and f be strictly convex in the range of u^0. That is, there exist constants $\gamma_1 \le \gamma_2$ such that*

$$0 < \gamma_1 \le f''(w) \le \gamma_2$$

for any $|w| \le \|u^0\|_{L^\infty}$. Then, there exists a constant $\kappa > 0$ which depends only on the ratio γ_1/γ_2 such that under the CFL condition

$$\lambda \|a\|_\infty \le \kappa \tag{16}$$

the l_2 norm of the nonnegative jumps of the NT scheme is non-increasing in time. That is, the NT scheme satisfies the following one-sided Lipschitz condition

$$\sum_{j\in\mathbb{Z}} \left(\delta_j^{n+1}\right)_+^2 \le \sum_{j\in\mathbb{Z}} \left(\delta_j^n\right)_+^2, \tag{17}$$

for all $n \ge 0$, where we use the standard $+$ notation: $x_+ = \max(x, 0)$.

Proof It is enough to prove the result for one time step, assuming that $\sum_{j\in\mathbb{Z}} (\delta_j^n)_+^2 < \infty$. We will always assume that the CFL condition (16) is satisfied with $\kappa \le 0.32$ because this guarantees the TVD property of the NT scheme, see [18]. We will first prove the stability estimate (17) for an arbitrary nondecreasing sequence. Proving the general result will be the last step of the proof and follows from a localization argument similar to the one in [13].

Hence, we assume that all jumps δ_j^n are nonnegative. There are two different choices for f_j', see (10), and the proof is very similar for either one. We are only going to consider the second one here

$$f_j' = a(v_j^n)\, v_j'. \tag{18}$$

We use (9) and (18) in (14) and derive the formula for the new jumps $\delta_j' := \delta_j^{n+1}$ from the old ones $\delta_j = v_j^n - v_{j-1}^n$. That is

$$\begin{aligned}\delta_{j+1}' = &\frac{1}{2}(\delta_j + \delta_{j+1}) - \frac{1}{8}(y_{j+1} - 2y_j + y_{j-1}) \\ &- \lambda\left[f\left(v_{j+1}^{n+1/2}\right) - 2f\left(v_j^{n+1/2}\right) + f\left(v_{j-1}^{n+1/2}\right)\right],\end{aligned}$$

where we define $y_j := v'_j = \min(\delta_j, \delta_{j+1})$, see (9). Now, we consider the flux difference

$$f\left(v_{j+1}^{n+1/2}\right) - f\left(v_j^{n+1/2}\right) = \bar{a}_{j+1/2}\left(v_{j+1}^{n+1/2} - v_j^{n+1/2}\right),$$

where $\bar{a}_{j+1/2}$ is some averaged velocity. We will use the notation $a_j := a(v_j^n)$ and the standard divided difference notation $\bar{a}_{j+1/2} = f[v_{j+1}^{n+1/2}, v_j^{n+1/2}]$. Note that

$$v_{j+1}^{n+1/2} - v_j^{n+1/2} = \delta_{j+1} - \frac{\lambda}{2}(a_{j+1}y_{j+1} - a_j y_j). \tag{19}$$

Then

$$\begin{aligned}\delta'_{j+1} = &\frac{1}{2}(\delta_j + \delta_{j+1}) - \frac{1}{8}(\Delta y_{j+1} - \Delta y_j)\\ &- \lambda\left[\bar{a}_{j+1/2}\left(\delta_{j+1} - \frac{\lambda}{2}(a_{j+1}y_{j+1} - a_j y_j)\right)\right.\\ &\left. - \bar{a}_{j-1/2}\left(\delta_j - \frac{\lambda}{2}(a_j y_j - a_{j-1}y_{j-1})\right)\right],\end{aligned}$$

where we define $\Delta y_j := y_j - y_{j-1}$. Assuming $\kappa \le 1/4$, we derive

$$\delta'_{j+1} \le \left(\frac{1}{2} + \frac{1}{8} + \kappa + \kappa^2\right)(\delta_j + \delta_{j+1}) \le \delta_j + \delta_{j+1}. \tag{20}$$

Our goal is to show that the l_2 norm of the jumps decreases in time. That is, $\sum(\delta'_j)^2 \le \sum(\delta_j)^2$ for the NT scheme with a CFL condition $\lambda\|a\|_\infty \le \kappa$ with sufficiently small but fixed κ. We are going to follow closely the steps in our proof in the case of linear flux, see [20]. Unfortunately, in the case of strictly convex flux the formula for $\{\delta'_j\}$ is more complicated. For this proof to work, we need to control certain new terms due to the nonlinearity of the problem, while estimating other terms to prevent the proof from becoming impalpable.
We break the rest of the proof into eleven steps.

Step 1 Replacing δ'_j by a simpler quantity.

First, we are going to replace the term $a_{j+1}y_{j+1} - a_j y_j$ with $\bar{a}_{j+1/2}\Delta y_{j+1}$, and the term $a_j y_j - a_{j-1}y_{j-1}$ with $\bar{a}_{j-1/2}\Delta y_j$ in the the formula for δ'_{j+1} above. Let

$$\sigma_{j+1} := \bar{a}_{j+1/2}\left((a_{j+1}y_{j+1} - a_j y_j) - \bar{a}_{j+1/2}\Delta y_{j+1}\right)$$

and

$$\begin{aligned}\delta''_{j+1} = &\frac{1}{2}(\delta_j + \delta_{j+1}) - \frac{1}{8}(\Delta y_{j+1} - \Delta y_j)\\ &- \lambda\left[\bar{a}_{j+1/2}\left(\delta_{j+1} - \frac{\lambda}{2}\bar{a}_{j+1/2}\Delta y_{j+1}\right) - \bar{a}_{j-1/2}\left(\delta_j - \frac{\lambda}{2}\bar{a}_{j-1/2}\Delta y_j\right)\right].\end{aligned}$$

Again, assuming $\kappa \le 1/4$, it is easy to derive that

$$\delta''_{j+1} \le \delta_j + \delta_{j+1}. \tag{21}$$

We have that

$$\delta'_{j+1} - \delta''_{j+1} = \frac{\lambda^2}{2}(\sigma_{j+1} - \sigma_j).$$

Using $|\bar{a}_{j-1/2}| \le \|a\|_\infty$, we get

$$|\sigma_j| \le \|a\|_\infty \left| y_j(a_j - \bar{a}_{j-1/2}) - y_{j-1}(a_{j-1} - \bar{a}_{j-1/2}) \right|.$$

Note that

$$a_j - \bar{a}_{j-1/2} = a(v_j^n) - f\left[v_j^{n+1/2}, v_{j-1}^{n+1/2}\right] = a(v_j^n) - a(\xi_j^n),$$

where $\xi_j^n \in [v_{j-1}^{n+1/2}, v_j^{n+1/2}]$ from standard properties of divided differences. Assuming the CFL condition $\lambda\|a\|_\infty \le 0.5$, we have

$$v_j^{n+1/2} = v_j^n - \frac{\lambda a_j}{2} y_j \le v_j^n + \frac{\delta_j}{4} =: v_j^+,$$

and

$$v_{j-1}^{n+1/2} = v_{j-1}^n - \frac{\lambda a_{j-1}}{2} y_{j-1} \ge v_{j-1}^n - \frac{\delta_{j-1}}{4} =: v_{j-1}^-.$$

The above bounds and the Mean Value Theorem give

$$|a_j - \bar{a}_{j-1/2}| \le \max_{w \in [v_{j-1}^-, v_j^+]} |a'(w)| \max(v_j^n - v_{j-1}^-, v_j^+ - v_j^n) \le \gamma_2(\delta_j + \delta_{j-1}). \tag{22}$$

Similarly, we obtain

$$|a_{j-1} - \bar{a}_{j-1/2}| \le \gamma_2(\delta_j + \delta_{j-1}).$$

Using the above estimates and $y_j = \min(\delta_j, \delta_{j+1})$, we derive

$$|\sigma_j| \le 2\|a\|_\infty \gamma_2 \delta_j(\delta_j + \delta_{j-1}) \quad \text{and}$$
$$\left|\delta'_{j+1} - \delta''_{j+1}\right| \le 4\lambda^2 \|a\|_\infty \gamma_2 \, \max\left(\delta_{j+1}^2, \delta_j^2, \delta_{j-1}^2\right).$$

Using the above estimates, (20) and (21), we conclude

$$\left| \sum \left(\delta'_j\right)^2 - \sum \left(\delta''_j\right)^2 \right| \le 48\|a\|_\infty \gamma_2 \lambda^2 \sum (\delta_j)^3. \tag{23}$$

We now shift the index ($j := j+1$) in the definition of δ''_j and regroup the terms the following way

$$\delta''_j = \left(\frac{1}{2} + \lambda \bar{a}_{j-3/2}\right)\delta_{j-1} + \left(\frac{1}{2} - \lambda \bar{a}_{j-1/2}\right)\delta_j - \frac{1}{8}(\Delta y_j - \Delta y_{j-1})$$
$$+ \frac{\lambda^2}{2}\left((\bar{a}_{j-1/2})^2 \Delta y_j - (\bar{a}_{j-3/2})^2 \Delta y_{j-1}\right).$$

Let $\alpha_j := 1/2 + \lambda \bar{a}_{j-1/2}$ and $\varphi_j := \alpha_j(1-\alpha_j)$. With this notation, we have

$$\delta''_j = \alpha_{j-1}\delta_{j-1} + (1-\alpha_j)\delta_j - \frac{1}{2}\left(\varphi_j \Delta y_j - \varphi_{j-1}\Delta y_{j-1}\right). \qquad (24)$$

The above formula is close to the formula for the new jumps in the linear case [20], which is

$$\delta'_j = \alpha\delta_{j-1} + (1-\alpha)\delta_j - \frac{1}{2}\alpha(1-\alpha)\left(\Delta y_j - \Delta y_{j-1}\right).$$

Let $\mathcal{D} := \sum \delta_j^2 - \sum (\delta''_j)^2$.

Step 2 Rewriting $\mathcal{D}$ as $Q_1 + Q_2 + Q_3 +$ an error term.

Next, we proceed similarly to the linear case. There we represent $\mathcal{D}$ as sum of two certain quantities Q_1 and Q_2. Here we will have two similar quantities and a new quantity Q_3 due to the nonlinearity of the problem.
We have the following representation

$$\mathcal{D} = \sum \delta_j^2 - \sum (\delta''_j)^2 = I_1 + I_2 + I_3, \qquad (25)$$

where

$$\begin{aligned}
I_1 &= \sum \delta_j^2 - \sum (\alpha_{j-1}\delta_{j-1} + (1-\alpha_j)\delta_j)^2, \\
I_2 &= \sum (\alpha_{j-1}\delta_{j-1} + (1-\alpha_j)\delta_j)(\varphi_j\Delta y_j - \varphi_{j-1}\Delta y_{j-1}), \\
I_3 &= -\frac{1}{4}\sum (\varphi_j\Delta y_j - \varphi_{j-1}\Delta y_{j-1})^2
\end{aligned}$$

We transform the first term I_1 in the following way

$$\begin{aligned}
I_1 &= \sum \delta_j^2 - \sum \left(\alpha_{j-1}^2\delta_{j-1}^2 + 2\alpha_{j-1}(1-\alpha_j)\delta_{j-1}\delta_j + (1-\alpha_j)^2\delta_j^2\right) \\
&= \sum \left(\delta_j^2(1-\alpha_j^2 - (1-\alpha_j)^2) - 2\alpha_{j-1}(1-\alpha_j)\delta_{j-1}\delta_j\right) \\
&= \sum \left(\alpha_j(1-\alpha_j)\delta_j^2 + \alpha_{j-1}(1-\alpha_{j-1})\delta_{j-1}^2 - 2\alpha_{j-1}(1-\alpha_j)\delta_{j-1}\delta_j\right) \\
&= \sum \alpha_{j-1}(1-\alpha_j)(\delta_j - \delta_{j-1})^2 + \sum (\alpha_j - \alpha_{j-1})\left(\alpha_{j-1}\delta_{j-1}^2 + (1-\alpha_j)\delta_j^2\right)
\end{aligned}$$

Therefore, we have

$$I_1 = \sum \alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 + \sum \Delta\alpha_j\left(\alpha_{j-1}\delta_{j-1}^2 + (1-\alpha_j)\delta_j^2\right) \qquad (26)$$

Recall that

$$\varphi_j = \alpha_j(1-\alpha_j) = \left(\frac{1}{2} + \lambda\bar{a}_{j-1/2}\right)\left(\frac{1}{2} - \lambda\bar{a}_{j-1/2}\right). \qquad (27)$$

Similar to (22), we derive

$$|\Delta\varphi_j| = \lambda^2\left|(\bar{a}_{j-3/2})^2 - (\bar{a}_{j-1/2})^2\right| \le 3\lambda^2\|a\|_\infty\gamma_2(\delta_{j-1} + \delta_j). \qquad (28)$$

The above estimate will help us replace φ_j with φ_{j-1} in I_3. Define

$$I_3' := -\frac{1}{4}\sum\left(\varphi_{j-1}\Delta^2 y_j\right)^2, \tag{29}$$

and let

$$E_1 = I_3 - I_3'.$$

Using (28) and (29) we get

$$|E_1| \le 6\lambda^2\|a\|_\infty\gamma_2\sum(\delta_j)^3.$$

We use (29) in (25) and split $\mathcal{D}$ in the following way

$$\mathcal{D} = Q_1 + Q_2 + Q_3 + E_1, \tag{30}$$

where

$$Q_1 := \sum\alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 + I_2 - \frac{1}{2}\sum(\varphi_{j-1}\Delta^2\delta_j)^2, \tag{31}$$

$$Q_2 := \frac{1}{4}\left[2\sum(\varphi_{j-1}\Delta^2\delta_j)^2 - \sum(\varphi_{j-1}\Delta^2 y_j)^2\right], \tag{32}$$

and

$$Q_3 := \sum\Delta\alpha_j\left(\alpha_{j-1}\delta_{j-1}^2 + (1-\alpha_j)\delta_j^2\right). \tag{33}$$

Again, if we take $\alpha_j = \alpha$ and denote $\beta := \frac{1}{2}\alpha(1-\alpha)$, we are going to get the split we used in [20] with Q_1 here equal to $2\beta Q_1$ in [20], Q_2 here equal to $\beta^2 Q_2$ in our notation from [20], and $Q_3 = 0$ in [20]. The new term Q_3 is due to the nonlinearity of the flux.

Step 3 A lower bound for Q_3.

Using that $\Delta\alpha_j = \lambda(\bar{a}_{j-1/2} - \bar{a}_{j-3/2})$, we derive

$$\begin{aligned}\Delta\alpha_j &= \lambda f\left[v_j^{n+1/2}, v_{j-1}^{n+1/2}, v_{j-2}^{n+1/2}\right]\left(v_j^{n+1/2} - v_{j-2}^{n+1/2}\right)\\ &= \frac{\lambda f''(\xi)}{2}\left(v_j^{n+1/2} - v_{j-2}^{n+1/2}\right)\end{aligned}$$

Under the CFL condition $\lambda\|a\|_\infty \le 0.5$, we have

$$v_j^{n+1/2} = v_j^n - \frac{\lambda}{2}a\left(v_j^n\right)y_j \ge v_j^n - \frac{\delta_j}{4}$$

and

$$v_{j-2}^{n+1/2} = v_{j-2}^n - \frac{\lambda}{2}a\left(v_{j-2}^n\right)y_{j-2} \le v_{j-2}^n + \frac{\delta_{j-1}}{4}.$$

Using the above inequalities, we obtain

$$\Delta\alpha_j \ge \frac{\lambda\gamma_1}{2}\left(v_j^{n+1/2} - v_{j-2}^{n+1/2}\right) \ge \frac{3}{8}\lambda\gamma_1(\delta_j + \delta_{j-1}), \tag{34}$$

where γ_1 is the minimum convexity of the flux. Hence, we derive a lower bound for the nonlinear term

$$Q_3 \geq \frac{1}{15}\gamma_1\lambda \sum(\delta_j)^3. \tag{35}$$

Note that, both $|\sum(\delta'_j)^2 - \sum(\delta''_j)^2|$ and $|E_1|$ are of order $O\left(\lambda^2 \sum(\delta_j)^3\right)$ and are be dominated by $\lambda \sum \delta_j^3$ for a sufficiently small λ. Using three lemmas (see Lemmas 2–4 in [20]), in the case of linear flux we proved that both terms Q_1 and Q_2 are nonnegative and their sum $\mathcal{D} = Q_1 + Q_2$ satisfies $\mathcal{D} \geq \frac{\beta^3}{4} \sum(\Delta^2\delta_j)^2$. Going through the same steps as in [20], we will prove the following lemma which concludes the proof of Theorem 1 in the case of non-negative jumps.

Lemma 2 *For any λ sufficiently small, we have*

$$\mathcal{D} = Q_1 + Q_2 + Q_3 + E_1 \geq C\left(\lambda \sum(\delta_j)^3 + \sum(\Delta^2\delta_j)^2\right) \tag{36}$$

with a constant C which depends only on the ratio γ_1/γ_2 and $\|a\|_\infty$.

Proof We will transform Q_1 and Q_2 in the form needed to use the lower bound in Lemma 1 from [20].

Step 4 Initial transformation of I_2.

We start with I_2. Using Abel summation we get

$$\begin{aligned} I_2 &= \sum \varphi_j \Delta y_j \left(\alpha_{j-1}\delta_{j-1} + (1-\alpha_j)\delta_j - \alpha_j\delta_j - (1-\alpha_{j+1})\delta_{j+1}\right) \\ &= -\sum \varphi_j \Delta y_j \left(\alpha_{j-1}\Delta\delta_j + (1-\alpha_{j+1})\Delta\delta_{j+1} + \delta_j(\Delta\alpha_j - \Delta\alpha_{j+1})\right). \end{aligned}$$

We split I_2 in two parts

$$\begin{aligned} I_2 = &-\sum \varphi_j \Delta y_j \left(\alpha_{j-1}\Delta\delta_j + (1-\alpha_{j+1})\Delta\delta_{j+1}\right) \\ &+ \sum \varphi_j \Delta y_j \delta_j \left(\Delta\alpha_{j+1} - \Delta\alpha_j\right). \end{aligned}$$

Using the above in (31), we get

$$\begin{aligned} Q_1 = &\sum \alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 + \sum \varphi_j \Delta y_j \delta_j \left(\Delta\alpha_{j+1} - \Delta\alpha_j\right) \\ &- I''_2 - \frac{1}{2}\sum(\varphi_{j-1}\Delta^2\delta_j)^2, \end{aligned} \tag{37}$$

where

$$I''_2 := \sum \varphi_j \Delta y_j \left(\alpha_{j-1}\Delta\delta_j + (1-\alpha_{j+1})\Delta\delta_{j+1}\right) = A + B, \tag{38}$$

and we define

$$A = \sum_j \varphi_j(1-\alpha_{j+1})\Delta\delta_{j+1}\Delta y_j \quad \text{and} \quad B = \sum_j \varphi_j\alpha_{j-1}\Delta\delta_j\Delta y_j.$$

We are going to split A and B in parts. We proceed exactly in the same way as in the linear case, see the proof of Lemma 2 in [20]. The new elements here are that we have $\sum_j \varphi_j$ instead of $\sum_j$ and the additional multipliers $(1-\alpha_{j+1})$ and α_{j-1} in each sum (in the linear case $\alpha_j = \alpha$ for all j). Replacing φ_j by φ_{j-1} in three of the resulting sums, and using (28) to estimate the resulting error term E_2, we get

$$\begin{aligned} A = E_2 &+ \sum_{\Delta\delta_j<0} \varphi_j(1-\alpha_j)(\Delta\delta_j)^2 + \sum_{\Delta\delta_j\ge 0} \frac{\varphi_j}{2}\left((1-\alpha_{j+1})+(1-\alpha_j)\right)(\Delta\delta_j)^2 \\ &- \frac{1}{2}\sum_{\Delta\delta_j\ge 0,\Delta\delta_{j+1}<0} \varphi_j(1-\alpha_{j+1})(\Delta\delta_j)^2 \\ &- \frac{1}{2}\sum_{\Delta\delta_j\ge 0,\Delta\delta_{j-1}<0} \varphi_j(1-\alpha_j)(\Delta\delta_j)^2 \\ &- \frac{1}{2}\sum_{\Delta\delta_{j-1}\ge 0,\Delta\delta_j\ge 0} \varphi_j(1-\alpha_j)(\Delta^2\delta_j)^2 \\ &+ \sum_{\Delta\delta_j\ge 0,\Delta\delta_{j+1}<0} \varphi_j(1-\alpha_{j+1})\Delta\delta_j\Delta\delta_{j+1}, \end{aligned} \tag{39}$$

where

$$|E_2| \le \sum |\Delta\varphi_j|((\Delta\delta_j)^2 + (\Delta^2\delta_j)^2) \le 90\lambda^2\|a\|_\infty\gamma_2\sum(\delta_j)^3.$$

Similarly,

$$\begin{aligned} B = E_3 &+ \sum_{\Delta\delta_j\ge 0} \varphi_j\alpha_{j-1}(\Delta\delta_j)^2 + \sum_{\Delta\delta_j<0} \frac{\varphi_j}{2}\left(\alpha_{j-1}+\alpha_{j-2}\right)(\Delta\delta_j)^2 \\ &- \frac{1}{2}\sum_{\Delta\delta_j<0,\Delta\delta_{j+1}\ge 0} \varphi_j\alpha_{j-1}(\Delta\delta_j)^2 - \frac{1}{2}\sum_{\Delta\delta_j<0,\Delta\delta_{j-1}\ge 0} \varphi_j\alpha_{j-2}(\Delta\delta_j)^2 \\ &- \frac{1}{2}\sum_{\Delta\delta_{j-1}<0,\Delta\delta_j<0} \varphi_j\alpha_{j-2}(\Delta^2\delta_j)^2 + \sum_{\Delta\delta_j\ge 0,\Delta\delta_{j+1}<0} \varphi_j\alpha_{j-1}\Delta\delta_j\Delta\delta_{j+1}. \end{aligned} \tag{40}$$

where

$$|E_3| \le \sum |\Delta\varphi_j|((\Delta\delta_j)^2 + (\Delta^2\delta_j)^2) \le 90\lambda^2\|a\|_\infty\gamma_2\sum(\delta_j)^3.$$

(Again, the error terms E_2 and E_3 result from replacing φ_{j-1} by φ_j.)

Step 5 Simplifying A and B.

We replace in all but in the first two sums appearing in A and B, the α_j by $1/2$ and the φ_j by $1/4$. To estimate the resulting error terms, note that (19) and (27), imply

$$\left|\alpha_j - \frac{1}{2}\right| \le \lambda\|a\|_\infty \quad \text{and} \quad \left|\varphi_j - \frac{1}{4}\right| \le \lambda^2\|a\|_\infty^2.$$

Furthermore, only the first two sums in each representation, see (39) and (40), are of order $\sum_j(\Delta\delta_j)^2$. All other sums either involve parts of $\sum_j(\Delta^2\delta_j)^2$ or the summation $\sum_{j\in\Lambda}(\Delta\delta_j)^2$ is over an index set Λ which is determined by two consecutive first differences with different signs: $\Delta\delta_j$ and $\Delta\delta_{j-1}$; or $\Delta\delta_j$ and $\Delta\delta_{j+1}$. Therefore, in all such cases either $(\Delta\delta_j)^2 \le (\Delta^2\delta_j)^2$ or $(\Delta\delta_j)^2 \le (\Delta^2\delta_{j+1})^2$. Replacing the α_j and the φ_j in A and B in the way indicated above we obtain,

$$\begin{aligned} A = &\sum_{\Delta\delta_j<0} \varphi_j(1-\alpha_j)(\Delta\delta_j)^2 + \sum_{\Delta\delta_j\ge 0} \frac{\varphi_j}{2}\left((1-\alpha_{j+1})+(1-\alpha_j)\right)(\Delta\delta_j)^2 \\ &-\frac{1}{2}\sum_{\Delta\delta_j\ge 0,\Delta\delta_{j+1}<0}\frac{1}{8}(\Delta\delta_j)^2 - \frac{1}{2}\sum_{\Delta\delta_j\ge 0,\Delta\delta_{j-1}<0}\frac{1}{8}(\Delta\delta_j)^2 \\ &-\frac{1}{2}\sum_{\Delta\delta_{j-1}\ge 0,\Delta\delta_j\ge 0}\frac{1}{8}(\Delta^2\delta_j)^2 + \sum_{\Delta\delta_j\ge 0,\Delta\delta_{j+1}<0}\frac{1}{8}\Delta\delta_j\Delta\delta_{j+1} + E_4, \end{aligned} \quad (41)$$

where

$$|E_4| \le |E_2| + \frac{1}{2}\lambda\|a\|_\infty \sum(\Delta^2\delta_j)^2.$$

Similarly,

$$\begin{aligned} B = &\sum_{\Delta\delta_j\ge 0} \varphi_j\alpha_{j-1}(\Delta\delta_j)^2 + \sum_{\Delta\delta_j<0} \frac{\varphi_j}{2}\left(\alpha_{j-1}+\alpha_{j-2}\right)(\Delta\delta_j)^2 \\ &-\frac{1}{2}\sum_{\Delta\delta_j<0,\Delta\delta_{j+1}\ge 0}\frac{1}{8}(\Delta\delta_j)^2 - \frac{1}{2}\sum_{\Delta\delta_j<0,\Delta\delta_{j-1}\ge 0}\frac{1}{8}(\Delta\delta_j)^2 \\ &-\frac{1}{2}\sum_{\Delta\delta_{j-1}<0,\Delta\delta_j<0}\frac{1}{8}(\Delta^2\delta_j)^2 + \sum_{\Delta\delta_j\ge 0,\Delta\delta_{j+1}<0}\frac{1}{8}\Delta\delta_j\Delta\delta_{j+1} + E_5 \end{aligned} \quad (42)$$

where

$$|E_5| \le |E_3| + \frac{5}{8}\lambda\|a\|_\infty \sum(\Delta^2\delta_j)^2.$$

Step 6 Simplifying and rearranging Q_1.

We replace φ_{j-1} by $1/4$ in the last sum of (37). This results in a new error term E_6 and we have

$$|E_6| \le \frac{1}{8}\lambda\|a\|_\infty \sum(\Delta^2\delta_j)^2.$$

Then, we use (41) and (42) in (38) and the modified (37), and obtain

$$Q_1 = R_1 + Q_1^* - E_4 - E_5 - E_6, \quad (43)$$

where

$$\begin{aligned} R_1 := &\sum \alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 + \sum \varphi_j\Delta y_j\delta_j\left(\Delta\alpha_{j+1}-\Delta\alpha_j\right) \\ &-\sum_{\Delta\delta_j<0}\varphi_j(1-\alpha_j)(\Delta\delta_j)^2 - \sum_{\Delta\delta_j\ge 0}\frac{\varphi_j}{2}\left((1-\alpha_{j+1})+(1-\alpha_j)\right)(\Delta\delta_j)^2 \\ &-\sum_{\Delta\delta_j\ge 0}\varphi_j\alpha_{j-1}(\Delta\delta_j)^2 - \sum_{\Delta\delta_j<0}\frac{\varphi_j}{2}\left(\alpha_{j-1}+\alpha_{j-2}\right)(\Delta\delta_j)^2, \end{aligned} \quad (44)$$

and

$$\begin{aligned} Q_1^* := \frac{1}{8}\Bigg(&\frac{1}{2}\sum_{\Delta\delta_j\geq 0,\Delta\delta_{j+1}<0}(\Delta\delta_j)^2+\frac{1}{2}\sum_{\Delta\delta_j\geq 0,\Delta\delta_{j-1}<0}(\Delta\delta_j)^2\\ &+\frac{1}{2}\sum_{\Delta\delta_{j-1}\geq 0,\Delta\delta_j\geq 0}(\Delta^2\delta_j)^2-\sum_{\Delta\delta_j\geq 0,\Delta\delta_{j+1}<0}\Delta\delta_j\Delta\delta_{j+1}\\ &+\frac{1}{2}\sum_{\Delta\delta_j<0,\Delta\delta_{j+1}\geq 0}(\Delta\delta_j)^2+\frac{1}{2}\sum_{\Delta\delta_j<0,\Delta\delta_{j-1}\geq 0}(\Delta\delta_j)^2\\ &+\frac{1}{2}\sum_{\Delta\delta_{j-1}<0,\Delta\delta_j<0}(\Delta^2\delta_j)^2-\sum_{\Delta\delta_j\geq 0,\Delta\delta_{j+1}<0}\Delta\delta_j\Delta\delta_{j+1}.\Bigg)\\ &-\frac{1}{32}\sum(\Delta^2\delta_j)^2. \end{aligned} \tag{45}$$

Step 7 Simplifying Q_2, obtaining a lower bound for $Q_1^* + Q_2$.

As before, we replace φ_j with $1/4$ in Q_2, see (32). Denote the resulting error term by E_7. Then,

$$|E_7| \leq \frac{1}{16}\lambda\|a\|_\infty\left(2\sum(\Delta^2\delta_j)^2+\sum(\Delta^2 y_j)^2\right)$$

Now, Lemma 4 in [20] implies $\sum(\Delta^2 y_j)^2 \leq 2\sum(\Delta^2\delta_j)^2$ (and the Lemma applies in our case too, since our definition of y_j is consistent with [20]). Thus,

$$|E_7| \leq \frac{1}{4}\lambda\|a\|_\infty\sum(\Delta^2\delta_j)^2.$$

We have

$$Q_2 - E_7 = Q_2^* := \frac{1}{64}\left(2\sum(\Delta^2\delta_j)^2-\sum(\Delta^2 y_j)^2\right), \tag{46}$$

We now observe that $Q_1^*+Q_2^*$ is identical to the term $2\beta Q_1+\beta^2 Q_2$ from Lemma 4 in [20] with $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{8}$. We use the lower bound of that lemma and obtain

$$Q_1^* + Q_2^* \geq \frac{1}{2048}\sum(\Delta^2\delta_j)^2. \tag{47}$$

Step 8 A lower bound for R_1.

We need to find a lower bound for R_1. Applying Abel summation to the second term in (44) and replacing φ_{j-1} by φ_j in one of the resulting terms, we get

$$\begin{aligned} R_1 - E_8 = &\sum\alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2+\sum\varphi_j\Delta\alpha_j\left(\delta_{j-1}\Delta y_{j-1}-\delta_j\Delta y_j\right)\\ &-\sum_{\Delta\delta_j<0}\varphi_j\left(1-\alpha_j+\frac{\alpha_{j-1}+\alpha_{j-2}}{2}\right)(\Delta\delta_j)^2\\ &-\sum_{\Delta\delta_j\geq 0}\varphi_j\left(1+\alpha_{j-1}-\frac{\alpha_j+\alpha_{j+1}}{2}\right)(\Delta\delta_j)^2. \end{aligned} \tag{48}$$

where

$$|E_8| \le 6\lambda^2 \|a\|_\infty^2 \gamma_2 \sum (\delta_j)^3.$$

(E_8 is the error term resulting from replacing φ_{j-1} by φ_j in the first term of the second sum in (48).)

Recall that $\Delta\alpha_j = \alpha_j - \alpha_{j-1} \ge 0$, see (34). Replacing α_{j-2} by α_{j-1} in the third sum of (48), and α_{j+1} by α_j in the fourth sum we get

$$\begin{aligned} R_1 - E_8 &\ge \sum \varphi_j \Delta\alpha_j \left(\delta_{j-1}\Delta y_{j-1} - \delta_j \Delta y_j\right) \\ &\quad + \sum \alpha_{j-1}(1-\alpha_j)(\Delta\delta_j)^2 - \sum \varphi_j \left(1 - \Delta\alpha_j\right)(\Delta\delta_j)^2 \\ &= R_2 + R_3, \end{aligned} \tag{49}$$

where

$$R_2 := \sum \varphi_j \Delta\alpha_j \left(\delta_{j-1}\Delta y_{j-1} - \delta_j \Delta y_j\right),$$

and

$$R_3 := \sum \left(\alpha_{j-1}(1-\alpha_j) - \varphi_j(1-\Delta\alpha_j)\right)(\Delta\delta_j)^2.$$

Combining (30), (33), (43), (46), and (47), we derive

$$\sum \delta_j^2 - \sum (\delta_j'')^2 = \mathcal{D} \ge Q_3 + R_2 + R_3 + \frac{1}{2048} \sum (\Delta^2\delta_j)^2 + E_9, \tag{50}$$

where $E_9 = E_1 - E_4 - E_5 - E_6 + E_7 + E_8$.
Let $Q_3^* := Q_3 + R_2 + R_3$. Using $\varphi_j = \alpha_j(1-\alpha_j)$ we obtain

$$Q_3^* = Q_3 + R_2 - \sum (1-\alpha_j)^2 \Delta\alpha_j (\Delta\delta_j)^2.$$

Step 9 A lower bound for Q_3^*.

Similar to (34), we estimate $\Delta\alpha_j \le \frac{5}{8}\lambda\gamma_2(\delta_j + \delta_{j-1})$. Next, we replace φ_j with $1/4$ and α_j with $1/2$ in all terms of Q_3^* with a resulting change E_{10}. We have $|E_{10}| \le \lambda^2 \|a\|_\infty^2 \gamma_2 \sum (\delta_j)^3$. Therefore,

$$\begin{aligned} Q_3^* - E_{10} &= \frac{1}{4} \sum \Delta\alpha_j \left(2\delta_{j-1}^2 + 2\delta_j^2 + \delta_{j-1}\Delta y_{j-1} - \delta_j \Delta y_j - (\Delta\delta_j)^2\right) \\ &= \frac{1}{4} \sum \Delta\alpha_j z_j, \end{aligned} \tag{51}$$

where

$$z_j := 2\delta_{j-1}^2 + 2\delta_j^2 + \delta_{j-1}\Delta y_{j-1} - \delta_j \Delta y_j - (\Delta\delta_j)^2.$$

Using that $\Delta y_j = y_j - y_{j-1}$, $\Delta\delta_j = \delta_j - \delta_{j-1}$, and $y_j = \min(\delta_j, \delta_{j+1}) \ge 0$, we derive

$$\begin{aligned} z_j &\ge (\delta_{j-1} + \delta_j)^2 - \delta_{j-1} y_{j-2} - \delta_j y_j + y_{j-1}(\delta_{j-1} + \delta_j) \\ &\ge 2\delta_{j-1}\delta_j + y_{j-1}(\delta_{j-1} + \delta_j). \end{aligned}$$

Now, we claim that

$$\Delta\alpha_j z_j + \Delta\alpha_{j-1} z_{j-1} \geq \frac{3}{4}\gamma_1\lambda(\delta_{j-1})^3. \tag{52}$$

There are three cases to consider:

Case 1 $\delta_j \geq \delta_{j-1}$. In this case $y_{j-1} = \delta_{j-1}, z_j \geq 4\delta_{j-1}^2$ and using (34), we obtain $\Delta\alpha_j z_j \geq \frac{9}{4}\gamma_1\lambda(\delta_{j-1})^3$.

Case 2 $\delta_{j-2} \geq \delta_{j-1}$. Similarly, $y_{j-2} = \delta_{j-1}$, $z_{j-1} \geq 4\delta_{j-1}^2$ and using (34), we obtain $\Delta\alpha_{j-1} z_{j-1} \geq \frac{9}{4}\gamma_1\lambda(\delta_{j-1})^3$.

Case 3 $\delta_{j-1} > \delta_j$ and $\delta_{j-1} > \delta_{j-2}$. In this case $y_{j-2} = \delta_{j-2}$ and $z_j \geq \delta_{j-1}^2 - \delta_{j-1}\delta_{j-2}$. Thus, $\Delta\alpha_j z_j \geq \frac{3}{8}\gamma_1\lambda(\delta_{j-1}^3 - \delta_{j-1}^2\delta_{j-2})$. Moreover, $z_{j-1} \geq 2\delta_{j-2}\delta_{j-1}$, so $\Delta\alpha_{j-1} z_{j-1} \geq \frac{3}{4}\gamma_1\lambda\delta_{j-1}^2\delta_{j-2}$ proving the claim in this case, too.

Using (52) in (51), we conclude

$$Q_3^* - E_{10} \geq \frac{3}{32}\gamma_1\lambda\sum(\delta_j)^3. \tag{53}$$

Step 10 Assembling all pieces.

Combining (50), (53), and the estimates for the error terms E, we derive the lower bound

$$\begin{aligned}\mathcal{D} \geq\ & \frac{3}{32}\gamma_1\lambda\sum(\delta_j)^3 + \frac{1}{2048}\sum(\Delta^2\delta_j)^2 \\ & - 196\|a\|_\infty\gamma_2\lambda^2\sum(\delta_j)^3 - \frac{3}{2}\|a\|_\infty\lambda\sum(\Delta^2\delta_j)^2. \end{aligned} \tag{54}$$

Therefore, when

$$\lambda\|a\|_\infty \leq \min\left(\frac{1}{4000}, \frac{\gamma_1}{3000\gamma_2}\right),$$

we have

$$\mathcal{D} = \sum\delta_j^2 - \sum(\delta_j'')^2 \geq \frac{1}{36}\gamma_1\lambda\sum(\delta_j)^3 + \frac{1}{9000}\sum(\Delta^2\delta_j)^2. \tag{55}$$

This finishes the proof of Lemma 2. □

Using (23) we prove that

$$\sum\delta_j^2 - \sum(\delta_j')^2 \geq \frac{1}{265}\gamma_1\lambda\sum(\delta_j)^3 + \frac{1}{9000}\sum(\Delta^2\delta_j)^2 \geq 0 \tag{56}$$

provided

$$\lambda\|a\|_\infty \leq \frac{\gamma_1}{2000\gamma_2},$$

i.e., for small but fixed CFL bound κ in (16). This completes the proof of Theorem 1 in the case of nonnegative jumps.

Step 11 The proof of Theorem 1 in the general case (localization argument).

We need to show that the one-sided l_2 norms inequality (17) holds for *any* initial sequence $\{\delta_j\}$. Let $\{w_j\}$ be a generic sequence of cell averages and $\{\delta_j\}$ be its jump sequence. That is,

$$||\{(\delta'_j)_+\}||_{l_2} \leq ||\{(\delta_j)_+\}||_{l_2} \tag{57}$$

holds for *any* initial sequence $\{\delta_j\}$ with finite l_2 norm. Recall that we proved (57) for any monotone sequence. The proof here will follow our localization proof in the case of linear flux in [13]. We consider the sequence $\{w_j\}$ and restrict the index j to a maximal subset Λ_m on which the piecewise constant function w is monotone, recall that $\delta_j = w_j - w_{j-1}$. Given a sequence $\{w_j\}$, we can decompose it into monotone subsequences. This decomposition also gives a decomposition of the sequence $\{\delta_j\}$ into subsequences such that in each subsequence all jumps have the same sign (non-negative or non-positive). Note that in the case of a sequence with non-positive jumps we have a trivial inequality in (57). Without any limitations, we assume that the jumps $\{\delta_j\}$ are non-negative for all $l \leq j \leq r$, $\delta_{l-1} < 0$ and $\delta_{r+1} < 0$. That is, w_{l-1} is a local minimum and w_r is a local maximum of the piecewise constant function w. Let w^m be the following piecewise constant correction of w

$$w_j^m := \begin{cases} w_j, & \text{if } l \leq j \leq r, \\ w_{l-1}, & \text{if } j < l, \\ w_r, & \text{if } j > r. \end{cases} \tag{58}$$

Note that $\Lambda_m = \{j : l \leq j \leq r\}$ and the jumps sequence $\delta^m := \{\delta_j^m\}$ of w^m is given by

$$\delta_j^m := \begin{cases} w_j - w_{j-1}, & \text{if } l \leq j \leq r, \\ 0, & \text{otherwise.} \end{cases} \tag{59}$$

In the case of a non-increasing subsequence, we extend it with constant values analogous to (58). Hence, we have a sequence of monotone functions $\{w^m\}$ and the corresponding jump sequences $\{\delta^m\} := \{\delta_j^m\}_{j\in\mathbb{Z}}$ such that

$$\sum_m \sum_{j\in\mathbb{Z}} (\delta^m)_+^2 = \sum_m \sum_{j\in\Lambda_m} (\delta_j^m)_+^2 = \|\{(\delta_j)_+\}\|_{l_2}^2$$

because the sequence of the jumps of $\{\delta_j\}$ is decomposed into disjoint jump subsequences $\{\delta_j^m\}$. We only consider the nonnegative jumps because we are in the case of convex flux and the l_2 norm of the jumps decreases only for nondecreasing initial data. There are two types of jumps δ'_j. A jump δ'_j is of *type 1* if it is equal to the jump $\delta'_j(\delta^m)$ – that is the jump generated with the starting sequence δ^m, where the index m such that $j \in \Lambda_m$. A jump is of *type 2* if it is not of *type 1*. Note that a *type 2* jump δ'_{j*} occurs only inside an interval which contains a strict local extremum. Near a local extremum we have two new nonzero jumps, say $(\delta^l_{j*})'$ and $(\delta^f_{j*})'$, generated using the two monotone w^m-s with index sets finishing/starting with j^*. Let's consider the case of a strict local maximum of the sequence, for example take w_r. The jump of type 2 is δ'_{r+1} and the corresponding left and right jumps generated by the monotone sequences $\{w^m\}$ and $\{w^{m+1}\}$ are $(\delta^l_{r+1})'$ and $(\delta^f_{r+1})'$. The jump $(\delta^l_{r+1})'$ is the last positive jump generated by the nondecreasing sequence $\{w^m\}$ and the jump $(\delta^f_{r+1})'$ is the first negative jump generated by the non-increasing sequence $\{w^{m+1}\}$. It is easy to verify that

$$\delta'_{r+1} = \left(\delta^l_{r+1}\right)' + \left(\delta^f_{r+1}\right)'. \tag{60}$$

Hence, we have

$$(\delta'_{r+1})_+ \le \left(\left(\delta^l_{r+1}\right)'\right)_+ + \left(\left(\delta^f_{r+1}\right)'\right)_+ \tag{61}$$

because the jumps $(\delta^l_{r+1})'$ and $(\delta^r_{r+1})'$ have opposite signs. In the case of a strict local minimum we derive (60) and (61) in the same way. In the remaining case of a local extremum over more than one cell, there are no jumps of type 2. Therefore, we conclude that

$$\sum_j (\delta'_j)^2_+ \le \sum_m \sum_{j\in\Lambda_m} (\delta'_j(\delta^m))^2_+ \le \sum_m \sum_{j\in\Lambda_m} (\delta^m_j)^2_+ = \sum_j (\delta_j)^2_+,$$

where we use the notation $\delta'_j(\delta^m)$ for the new jumps generated by $\{\delta^m\}$. □

4 Convergence and error estimates

In this section we are going to use our onesided stability result, Theorem 1, to prove the convergence of the NT scheme to the entropy solution of

$$\begin{cases} u_t + f(u)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u^0(x), & x \in \mathbb{R}. \end{cases} \tag{62}$$

In [18], convergence is proven via a single cell entropy inequality. Unfortunately, in order to satisfy the inequality, the authors impose an additional restriction in all regions where the numerical solution is increasing. This reduces the formal order of the NT scheme in such regions to first order. They also note that the additional restriction is not necessary in the applications and one should use the true NT scheme for numerical computations. In order to describe the next result, we need to introduce some notation. A function g is of bounded variation, i.e., $g \in \mathrm{BV}(\mathbb{R})$, if

$$|g|_{\mathrm{BV}(\mathbb{R})} := \sup \sum_{i=1}^{n} |g(x_{i+1}) - g(x_i)| < \infty,$$

where the supremum is taken over all finite sequences $x_1 < \cdots < x_n$ in $\mathbb{R}$. Functions of bounded variation have at most countably many discontinuities, and their left and right limits $g(x^-)$ and $g(x^+)$ exist at each point $x \in \mathbb{R}$. Since the values of the initial condition u^0 on a set of measure zero have no influence on the numerical solution v and the entropy solution solution u, it is desirable to replace the semi-norm $|\cdot|_{\mathrm{BV}(\mathbb{R})}$ by a similar quantity independent of the function values on sets of measure zero. The standard approach in conservation laws is to consider the space Lip(1, $L^1(\mathbb{R})$) of all functions $g \in L^1(\mathbb{R})$ such that the seminorm

$$|g|_{\text{Lip}(1,L^1(\mathbb{R}))} := \limsup_{y>0} \frac{1}{y} \int_{\mathbb{R}} |g(x+y) - g(x)| \, \mathrm{d}x \tag{63}$$

is finite. It is clear that $|g|_{\text{Lip}(1,L^1(\mathbb{R}))}$ will not change if g is modified on a set of measure zero. At the same time the above two seminorms are equal for functions $g \in \text{BV}(\mathbb{R})$ such that the value of g at a point of discontinuity lies between $g(x^-)$ and $g(x^+)$ (see Theorem 9.3 in [5]). Similarly, we define the space Lip(s, $L^p(\mathbb{R})$), $1 \le p \le \infty$ and $0 < s \le 1$, which is the set of all functions $g \in L^p(\mathbb{R})$ for which

$$\|g(\cdot - y) - g(\cdot)\|_{L^p(\mathbb{R})} \le My^s, \quad y > 0. \tag{64}$$

The smallest $M \ge 0$ for which (64) holds is $|g|_{\text{Lip}(s,L^p(\mathbb{R}))}$. It is easy to see that in the case $p = 1$ and $s = 1$ the seminorm given in (64) is the same as the one in (63). In the case $p > 1$, the space Lip(1, $L^p(\mathbb{R})$) is essentially the same as $W^1(L^p(\mathbb{R}))$, see [5] for details. Because our stability result is onesided, for functions $g \in$ Lip(1, $L^1(\mathbb{R})$) we consider the classes Lip(s, L^p)+ defined by

$$\| \left(g(\cdot - y) - g(\cdot)\right)_+ \|_{L^p(\mathbb{R})} \le My^s, \quad y > 0. \tag{65}$$

The smallest $M \ge 0$ for which (65) holds is denoted by $|g|_{\text{Lip}(s,L^p)+}$. When we set $p = \infty$ and $s = 1$, we obtain the class Lip(1, L^∞)+ which is the usual onesided Lipschitz class used in conservation laws denoted by Lip+, see for example [19]. In our previous work [13], we proved that for any $u^0 \in$ Lip(1, L^2) the discrete l_2 norm of the jumps satisfies

$$\|\{\delta_j^0\}\|_2^2 = \sum_j (\delta_j^0)^2 \le h\|u^0\|^2_{\text{Lip}(1,L^2)},$$

see [13] for details. Similarly, it is easy to show the onesided analog

$$\sum_j (\delta_j^0)_+^2 \le h\|u^0\|^2_{\text{Lip}(1,L^2)+}. \tag{66}$$

We now use Theorem 1 and derive the following onesided bound

$$\sum_j (\delta_j^n)_+^2 \le \sum_j (\delta_j^0)_+^2 \le h\|u^0\|^2_{\text{Lip}(1,L^2)+} \tag{67}$$

for any $n = 0, 1, \ldots$. Using the estimate

$$\max_j (\delta_j^n) \le \left(\sum_j (\delta_j^n)_+^2 \right)^{1/2}$$

in (67), we derive the Onesided Lipschitz bound

$$\max_{n,j} (\delta_j^n) \le h^{1/2}\|u^0\|_{\text{Lip}(1,L^2)+}. \tag{68}$$

For piecewise smooth solutions, it is well known that the fractional bound (68) is enough to guarantee convergence of the numerical method to the entropy solution. In the general case of initial data $u^0 \in \mathrm{Lip}(1, L^1(\mathbb{R}))$, we refer the reader to [12] where the authors develop a convergence theory for numerical methods for conservation laws and Hamilton–Jacobi equations. They show that a function u is the entropy solution of the conservation law if and only if the primitive function $U(x,t) := \int_{\infty}^{x} u(x,t)\mathrm{d}x$ is the viscosity solution of the corresponding Hamilton–Jacobi equation, see [12] for details. Moreover, it is shown that a class of TVD numerical methods with a weak onesided bound on second differences converges to the unique viscosity solution. Let $V_i^n := h\sum_{j\le i} v_j^n$ be the point values of the primitive function of the numerical solution v. The estimate which we need to verify is

$$V_{i+k}^n + V_{i-k}^n - 2V_i^n \le C\,(kh)^{2-\alpha} \tag{69}$$

with $0 \le \alpha < 1$, see $(2.3)_\alpha$ in [12], and this should hold for all i, k, n with a fixed constant C. Recall that $\delta_j^n = v_j^n - v_{j-1}^n$. It is easy to derive

$$V_{i+k}^n + V_{i-k}^n - 2V_i^n = h\sum_{j=i+1}^{i+k}\ \sum_{s=j-k+1}^{j} \delta_s^n .$$

Then, we estimate

$$V_{i+k}^n + V_{i-k}^n - 2V_i^n \le kh \sum_{j=i-k+1}^{i+k} (\delta_s^n)_+ \le \sqrt{2}\,hk^{3/2}\left(\sum_{j=i-k+1}^{i+k} (\delta_s^n)_+^2\right)^{1/2},$$

where we applied Cauchy-Schwartz for the last inequality. Using (66) and (67) above, we conclude

$$V_{i+k}^n + V_{i-k}^n - 2V_i^n \le C\,(kh)^{3/2}$$

which gives $(2.3)_\alpha$ in [12] with $\alpha = 1/2$. Therefore, we have the following theorem.

Theorem 3 *Let* $u^0 \in \mathrm{Lip}(1, L^1(\mathbb{R})) \cap \mathrm{Lip}(1, L^2)+$. *Then, there exists* $\kappa > 0$ *such that under the CFL condition* $\lambda\|a\|_\infty \le \kappa$ *the NT scheme described in* (14) *converges to the unique entropy solution of* (62).

It should be possible to develop a theory for error estimates based on (67) for $u^0 \in \mathrm{Lip}(1, L^1(\mathbb{R})) \cap \mathrm{Lip}(s, L^p)+$, $s > 1/2$. But the results do not immediately follow from the existing theory and are out of the scope of this paper. Here, we will discuss the case of nondecreasing initial data only. In our proof of Theorem 1, we considered only one approximation of the flux, see (18), and noted that the proof for the other approximation of the flux in (10) is analogous. In fact, one can derive the inequality (23) and a perturbation formula similar to (24) for the NT scheme with exact evolution in time – that is, when we compute the integrals in (7) exactly. Then, one can argue the same way as here and prove a analog of Theorem 1 for the NT scheme with exact evolution in time. We leave the proof to the reader and note that Theorem 1 is not valid for first order in time approximations of the flux.

This implies that Theorem 1 is a *true* second order result. Using the stability result (69) for the NT scheme with exact evolution in time, we observe that the modified Minmod scheme introduced in [14] (see (E2) on page 1763 in [14]) is actually never modified if $\alpha' = 1/2$ and the constant C is chosen appropriately. Therefore, the error estimate in Theorem 2 from [14] is valid (with a parameter $\alpha = 1/2$) for the NT scheme with exact evolution in time.

Theorem 4 *Let* $u^0 \in \mathrm{Lip}(1, L^1(\mathbb{R})) \cap \mathrm{Lip}(1, L^2)+$ *be a nondecreasing function. Then, there exists* $\kappa > 0$ *such that under the CFL condition* $\lambda \|a\|_\infty \le \kappa$ *the NT scheme with exact evolution in time converges to the unique entropy solution of* (62) *and satisfies the error estimate*

$$\|u(\cdot, T) - v(\cdot, T)\|_{L^1(\mathbb{R})} \le C h^{1/4} |u^0|_{\mathrm{Lip}(1, L^1(\mathbb{R}))}.$$

We include the above error estimate only to show that it is possible to derive error estimates from our new onesided stability. A general estimate for initial data $u^0 \in \mathrm{Lip}(1, L^1(\mathbb{R})) \cap \mathrm{Lip}(1, L^2)+$ requires a modification of our arguments in [14] or the dual Lip′-Lip+ arguments in [19] and will be addressed elsewhere.

Acknowledgment The authors are grateful to Ronald DeVore for his support and to the anonymous reviewers for their suggestions and remarks. This study has been supported by the NSF DMS Grant nos. 0510650 and 9970455

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